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On supergravity solutions of branes in Melvin universes

Mohsen Alishahiha,[a] Batool Safarzadeh[b] and Hossein Yavartanoo[a]

[a] *Institute for Studies in Theoretical Physics and Mathematics (IPM)*
P.O. Box 19395-5531, Tehran, Iran
[b] *Department of physics, School of Science*
Tarbiat Modares University, P.O. Box 14155-4838, Tehran, Iran
alishah@theory.ipm.ac.ir, safarzadeh@theory.ipm.ac.ir, yavar@ipm.ir

Abstract: We study supergravity solutions of type-II branes wrapping a Melvin universe. These solutions provide the gravity description of non-commutative field theories with non-constant non-commutative parameter. Typically these theories are non-supersymmetric, though they exhibit some feature of their corresponding supersymmetric theories. An interesting feature of these non-commutative theories is that there is a critical length in the theory in which for distances larger than this length the effects of non-commutativity become important and for smaller distances these effects are negligible. Therefore we would expect to see this kind of non-commutativity in large distances which might be relevant in cosmology. We also study M5-brane wrapping on 11-dimensional Melvin universe and its descendant theories upon compactifying on a circle.

Keywords: AdS-CFT Correspondence, D-branes.

Contents

1. Introduction

AdS/CFT correspondence [1−3] have probably provided a powerful framework for understanding quantum gravity. In this framework a quantum mechanical system which includes gravity can be described by a lower dimensional quantum mechanical system without gravity. Having had gravity on one side of the duality one may then wonder if we can learn about quantum gravity by studying a quantum field theory which by now we have more control on it. Of course, the point would be how to identify the two sides of the correspondence, namely starting from given gravitional theory how to find the corresponding field theory dual. Without such an identification, it seems that although AdS/CFT correspondence has opened up a window to understand the quantum gravity better, practically it could not help us so much.

Fortunately string theory and different branes in string theory have been able to give us a practical way to proceed and in fact by now we know several examples of AdS/CFT correspondence where we almost know two sides of the duality. In general one could start from a given brane configuration in string theory and check if the theory on the worldvolume of this brane configuration decouples from the bulk gravity in an special limit (decoupling limit). If so, one then expect that string theory (gravity) on this particular background would be dual to the theory lives on the worldvolume of the brane configuration.

The simplest example is to start from Dp-brane in type-II string theories. It can be shown that the brane worldvolume theory decouples from the bulk gravity for $p < 6$ [4]. Therefore type-II string theories in the near horizon limit (decoupling limit) of Dp-brane provide a gravity description for $(p+1)$-dimensional gauge theory with 16 supercharges [5]. In other words one may use these SYM theories with 16 supercharges to study string theory/gravity on these particular backgrounds. This procedure has also been generalized for other branes like NS5-brane as well as M-theory branes. See for example [6].

Considering a single brane in string theory would probably mean that we are restricting ourselves in a small region of string theory moduli space. In generic point of string theory moduli space we would expect different low energy fields have non-zero expectation value. In particular in generic point we would expect to have a non-zero NS-NS B-field. Turning on a B-field on the D-brane worldvolume can be viewed, via AdS/CFT correspondence, as a perturbation of the worldvolume field theory by an operator of dimension 6. For example in the D3-brane case, from the four dimensional superconformal Yang-Mills theory point of view the bosonic part of this dimension 6 operator is given by [7, 8]

$$\mathcal{O}_{\mu\nu} = \frac{1}{2g_{\rm YM}^2}\,{\rm Tr}\left(F_{\mu\delta}F^{\delta\rho}F_{\rho\nu} - F_{\mu\nu}F^{\rho\delta}F_{\rho\delta} + 2F_{\mu\rho}\sum_{i=1}^{6}\partial_\nu\phi^i\partial^\rho\phi^i - \frac{1}{2}F_{\mu\nu}\sum_{i=1}^{6}\partial_\rho\phi^i\partial^\rho\phi^i\right)\,, \tag{1.1}$$

where $g_{\rm YM}$ is the SYM coupling, $F_{\mu\nu}$ is the $\mathrm{U}(N)$ field strength and $\phi^i,\ i = 1, \cdots, 6$ are the adjoint scalars. This deformed theory by the operator $\mathcal{O}_{\mu\nu}$ can be extended to a complete theory with a simple description which is non-commutative SYM theory.

In fact it has been shown in [9 − 11] that, when we turn on a constant B-field on the D-brane worldvolume, the low-energy effective worldvolume theory is modified to be a non-commutative Super-Yang-Mills (NCSYM) theory. Actually the worldvolume theory of N coincident Dp-branes in the presence of a B-field is found to be $\mathrm{U}(N)$ NCSYM theory [12].

As in the case of zero B-field, there exists a limit where the bulk modes decouple from the worldvolume non-commutative field theory [12]; we expect to have a correspondence between string theory on the curved background with B-field and non-commutative field theories. In other words we expect to have a holographic picture like AdS/CFT correspondence [1 − 3] for the non-commutative theories. In fact this issue has been investigated in several papers, including [13 − 22].

So far we have considered the cases where the B-field is turned on in some spatial directions along the brane worldvolume. One could also consider cases where B-field has one leg along the time direction. While space non-commutativity can be accommodated within field theory, space-time non-commutativity seems to require string theory for consistency [23 − 27]. The B-field could also be light-like [28 − 30]. One may also consider the worldvolume theory of a D-brane in the presence of non-zero B-field with one leg along the brane worldvolume and the other along the transverse directions to the brane. This brane configuration was studied in [31 − 37] where the twisted compactification was introduced. This twisted compactification leads us to introduce a new type of star product between the

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fields at the level of effective field theory. The corresponding field theory is called dipole field theory.

One could also extend this consideration for NS5-brane/M-theory branes when we have non-zero RR field/3-form. In fact different deformations of NS5-brane with non-zero RR fields lead to theories on the worldvolume of NS5 branes, whose excitations include light-open Dp branes (ODp) [27, 29, 38, 39].

So far we have considered those theories which can be arisen in the brane worldvolume when we have uniform B-field. One may also consider the cases where the B-field is not uniform. This could lead to non-commutative field theories where the non-commutative parameter is non-constant [40 − 50]. Several aspects of non-commutative field theories with non-constant non-commutative parameter (including time dependent B-field) have been studied in [51 − 57]. This is the aim of this paper to further study supergravity solution of type-II string theories in the presence of non-zero B-field which could provide the gravity description of non-commutative field theories with non-constant non-commutative parameter. We will also extend this study for NS5-brane as well as M5-brane.

A common feature of adding B-field in the string theory (gravity) side of AdS/CFT correspondence is that the corresponding field theory dual turns out to be a non-local theory. On the other hand if we are willing to understand quantum gravity better, one would probably need to go beyond the local field theory. Therefore studying of these non-local field theories could increase our knowledge about general properties of non-local field theories.

The organization of the paper is as follows. In section 2 we will review the non-commutative gauge theory and the way we can write an invariant action when the non-commutative parameter is non-constant. In section 3 we will obtain the supergravity solution of Dp-brane wrapping a Melvin universe. In section 4 we shall study the decoupling limit of the supergravity solutions we have found in section 3. These could provide the gravity description of non-commutative gauge theory with a non-constant parameter in various dimensions. In section 5 we will study type-II NS5-brane wrapping a Melvin universe which leads to new supergravity solutions of NS5-brane in the presence of different RR fields which depend on the brane worldvolume coordinates. In section 6 this procedure is generalized to M5-brane. By compactifying this solution and bringing it to type-IIA and then using a chain of T and S dualities we will generate new solutions representing Dp-brane in the presence of B-field with one leg along time direction. This is in fact the generalization of NCOS theories where the non-commutative parameter is non-constant. To complete our discussions we study the light-like deformation in section 7. The last section is devoted to discussions.

2. Non-commutative gauge theory with non-constant parameter

In this section following [57] we review the non-commutative gauge theory with non-constant parameter. We will consider a special case of non-commutativity which can be defined in the worldvolume of D3-brane wrapping a Melvin universe. Although the non-commutative parameter is not constant, one can still study the corresponding gauge theory using some kind of star product. Of course it cannot be a simple Moyal product we usually use in the non-commutative gauge theory when its parameter is constant. This is because, it cannot be used to construct a gauge invariant action, taking into account that differentiation does not respect the product rule with non-constant parameter.

Nevertheless it has been shown [57] that in the polar coordinates the non-commutative parameter can be taken to be a constant. In fact for a four dimensional space parameterizing by t, r, ϕ, x the non-commutative parameter could be taken constant which we denote it by $\theta^{\phi x}$. In this notation the star product is defined as

$$f\#g = e^{\frac{i\theta^{\phi x}}{2}(\partial_\phi\partial_{x'}-\partial_{\phi'}\partial_x)} f(t,r,\phi,x)f(t,r,\phi',x')|_{\phi=\phi',x=x'}. \tag{2.1}$$

In these coordinates, one can define a set of unit vector fields as $\partial_i = X_a^\mu\partial_\mu$. Explicitly we have

$$\partial_1 = \partial_t, \qquad \partial_2 = \partial_r, \qquad \partial_3 = \frac{1}{r}\partial_\phi, \qquad \partial_4 = \partial_x. \tag{2.2}$$

The $\#$ and $*$ products are not related by a change of coordinates, though can be related using an automorphism $R(f)$ [58]

$$R(f\#g) = R(f) * R(g), \tag{2.3}$$

where R in leading order is given by [44]

$$R(f) = f + \frac{4\pi^2\theta^2}{24} r\partial_r\partial_x^2 f + \mathcal{O}(\theta^3). \tag{2.4}$$

By making use of this automorphism one can define a new derivation, which respects the product rule, as follows

$$\delta_{X_a} f = R\partial R^{-1} f. \tag{2.5}$$

Using this notation it is now straightforward to write the action for the corresponding non-commutative gauge theory [57]

$$S = \frac{1}{2}\,\mathrm{Tr}\int \sqrt{G}G^{ab}G^{cd}F_{ac} * F_{bd}, \quad F_{ab} = \delta_{X_a}A_b - \delta_{X_b}A_a + igA_a * A_b - igA_b * A_a, \tag{2.6}$$

where $G_{ab} = g_{\mu\nu}X_a^\mu X_b^\nu$. It can also be generalized for the case where we have scalar and spinor as well.

In the rest of this paper we shall study different aspects of non-commutative field theories defined by this non-commutative star product in various dimensions using their gravity duals.

3. The supergravity solution

In this section we shall study the supergravity solution of Dp-brane wrapped in a Melvin universe in type-II string theories.[1] This will lead, upon taking decoupling limit, to a non-commutative gauge theory on the brane worldvolume with non-constant non-commutativity. To find this supergravity solution we start from the supergravity solution of Dp-brane and performing a chain of T-dualities and twists. In fact the procedure is very similar

[1] D-brane in Melvin universe has been studied in [59 – 61]. The supergravity solutions of these kinds have also been studied in [62, 63].

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to one that studied [37] (see also [64]) in the context of dipole field theory and we shall follow its notation.

The supergravity solution of N coincident extremal Dp-branes in type-II string theories in string frame is given by [65]

$$
\begin{aligned}
ds^2 &= f^{-1/2}\left(-dt^2+\sum_{i=1}^{p-1}dx_i^2+dx_p^2\right)+f^{1/2}\left(d\rho^2+\rho^2 d\Omega_{8-p}^2\right),\\
e^{2\phi} &= g_s^2 f^{(3-p)/2},\qquad f=1+\frac{(2\pi)^{p-2}c_pNg_sl_s^{7-p}}{\rho^{7-p}},\qquad C_{01\cdots p}=-\frac{1}{g_s}f^{-1},
\end{aligned}
\tag{3.1}
$$

where $c_p=2^{7-2p}\pi^{(9-3p)/2}\Gamma(\frac{7-p}{2})$.

Suppose x_p is compact with radius β_p. Setting $x_p=\beta_p\theta$ and performing a T-duality along θ direction, one gets

$$
\begin{aligned}
ds^2 &= f^{-1/2}\left(-dt^2+\sum_{i=1}^{p-1}dx_i^2\right)+f^{1/2}\left(\frac{\alpha'^2}{\beta_p^2}d\tilde{\theta}^2+d\rho^2+\rho^2 d\Omega_{8-p}^2\right),\\
e^{2\phi} &= \frac{\alpha'^2}{\beta_p^2}g_s^2 f^{(4-p)/2},\qquad C_{0\cdots(p-1)}=-f^{-1},
\end{aligned}
\tag{3.2}
$$

which corresponds to $\mathrm{D}_{(p-1)}$-brane smeared along one direction, $\tilde{\theta}$, that is an angular coordinate with period $\tilde{\theta}\sim\tilde{\theta}+2\pi$. Now let us add a twist to the direction along the brane worldvolume as we go around the circle $\tilde{\theta}$

$$
dx_i\to dx_i+\sum_j\Omega_{ij}x_jd\tilde{\theta},
\tag{3.3}
$$

where Ω_{ij} is an element of Lie algebra $so(p-1)$. Therefore the metric changes to

$$
ds^2=f^{-1/2}\left(-dt^2+\sum_{i=1}^{p-1}\left(dx_i+\Omega_{ij}x_jd\tilde{\theta}\right)^2\right)+f^{1/2}\left(\frac{\alpha'^2}{\beta_p^2}d\tilde{\theta}^2+d\rho^2+\rho^2 d\Omega_{8-p}^2\right).
\tag{3.4}
$$

It is useful to set a new notation in which $X^T=(x_1,\cdots,x_{p-1})$ and therefore the above metric reads

$$
\begin{aligned}
ds^2 &= f^{-1/2}\left(-dt^2+dX^TdX+2(X^T\Omega^TdX)d\tilde{\theta}\right)+\left(f^{-1/2}X^T\Omega^T\Omega X+f^{1/2}\frac{\alpha'^2}{\beta_p^2}\right)d\tilde{\theta}^2\\
&+ f^{1/2}\left(d\rho^2+\rho^2 d\Omega_{8-p}^2\right).
\end{aligned}
\tag{3.5}
$$

Finally, once again, we can apply another T-duality on $\tilde{\theta}$ direction. Doing so, in the limit of $\beta_p\to\infty$ while keeping $\beta_p\Omega=M$ and $x_p=\beta_p\theta$ fixed, one finds

$$
\begin{aligned}
ds^2 &= f^{-1/2}\left(-dt^2+dr^2+r^2dn^Tdn+\frac{\alpha'^2dx_p^2-r^4f^{-1}\left(n^TMdn\right)^2}{\alpha'^2+r^2f^{-1}n^TM^TMn}\right)\\
&\quad +f^{1/2}\left(d\rho^2+\rho^2 d\Omega_{8-p}^2\right),\\
e^{2\phi} &= \frac{\alpha'^2g_s^2f^{(3-p)/2}}{\alpha'^2+r^2f^{-1}n^TM^TMn},\qquad \sum_i B_{pi}dx_i=\frac{r^2f^{-1}dn^TMn}{\alpha'^2+r^2f^{-1}n^TM^TMn}.
\end{aligned}
\tag{3.6}
$$

We will also get several RR fields which we have not written them here. We will write their explicit form when we consider each case in detail. Here we have also used a notation in which $X = rn$ for $n^T n = 1$.

It is worth noting that given a general supergravity solution of a system of branes, it is not clear whether the solution would give a well-defined description of some field theories. In fact, we must check to see whether there is a well-defined field theory on the brane worldvolume which decouples from bulk gravity. This can be done by evaluating the graviton absorption cross section. If there is a limit where graviton absorption cross section vanishes, we have a field theory which decouples from bulk gravity. Alternatively, one could calculate the potential that the gravitons feel because of the brane. Having a decoupled theory can be seen from the shape of the potential in the decoupling limit. Actually, for those branes which their worldvolume decouple from gravity, the potential develops an infinite barrier separating the space into two parts: bulk and brane. In this case the bulk's modes can not reach the brane because of this infinite barrier, and the same for brane's modes. Therefore the theory on the brane decouples from the bulk.

For the case we are interested in, perturbing the metric of the background (3.6), one finds the following equation for transverse gravitons [4]

$$\partial_\mu \left(\sqrt{-g} e^{-2\phi} g^{\mu\nu} \partial_\nu \Phi \right) = 0, \tag{3.7}$$

with $\Phi = h(r) e^{ik_\mu x^\mu}$. From this equation we can read the potential by writing it in the form of a Schrödinger-like equation as follows

$$\partial_\rho^2 \psi(\rho) + V_p(\rho)\psi(\rho) = 0 \ , \tag{3.8}$$

where the potential is given in terms of the metric components and in general it is messy to write the potential explicitly, though for the special case where the twist acts just on two directions one can write it in a simple closed form which can give us an insight whether the theory decouples. Doing so one arrives at

$$V_p(\rho) = -\left(1 + \frac{c_p N g_s (\omega l_s)^{7-p}}{\rho^{7-p}} \right) + \frac{(8-p)(6-p)}{4\rho^2}, \tag{3.9}$$

with $\rho = \omega r$. This potential is the same as the ordinary D-branes as well as branes in the presence of uniform B-field. Therefore we conclude that we have a decoupled theory living on the worldvolume of Dp-brane for $p \leq 5$. Although we have not written the potential for the most general twist, one can still show that in general the potential develops a barrier in the decoupling limit and thus we get decoupled theory.

4. Supergravity description of non-commutative gauge theory with non-constant parameter

In the previous section we have shown that the worldvolume theory of Dp-brane in the presence of non-zero B-field given in (3.6) decouples from the gravity for $p \leq 5$. Therefore the solution (3.6) can provide a dual gravity description for non-commutative field theory with non-constant parameter via AdS/CFT correspondence.

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For this supergravity solution the decoupling limit is defined as a limit in which $\alpha' \to 0$ and keeping the following quantities fixed

$$U = \frac{\rho}{l_s^2}, \qquad \bar{g_s} = g_s l_s^{p-3}. \tag{4.1}$$

In this limit the supergravity solution (3.6) reads

$$\begin{aligned} l_s^{-2} ds^2 &= h^{1/2} \left(-dt^2 + dr^2 + r^2 dn^T dn + \frac{dx_p^2 - r^4 h (n^T M dn)^2}{1 + r^2 h (n^T M^T M n)} \right) + h^{-1/2} \left(dU^2 + U^2 d\Omega_{8-p}^2 \right), \\ e^{2\phi} &= \frac{\bar{g}_s^2 h^{(p-3)/2}}{1 + r^2 h (n^T M^T M n)}, \qquad \sum_i B_{pi} dx_i = \frac{r^2 h (n^T M^T dn)}{1 + r^2 h (n^T M^T M n)}, \end{aligned} \tag{4.2}$$

where

$$R^{7-p} = 2^{7-2p} \pi^{(9-3p)/2} \Gamma\left(\frac{7-p}{2}\right) g_{YM}^2 N, \quad g_{YM}^2 = (2\pi)^{p-2} \bar{g_s}, \quad h = \left(\frac{U}{R}\right)^{7-p}. \tag{4.3}$$

The conjecture is now that the string theory on these backgrounds provides the gravity description of non-commutative gauge theories with non-constant non-commutative parameter in various dimensions.

The effective dimensionless coupling constant in the corresponding non-commutative field theory can be defined as $g_{\text{eff}}^2 \sim g_{\text{YM}}^2 N U^{p-3}$ and the scalar curvature of the metric in (4.2) has the behavior $l_s^2 \mathcal{R} \sim \frac{1}{g_{\text{eff}}}$. Thus the perturbative calculations in non-commutative field theory can be trusted when $g_{\text{eff}} \ll 1$, while when $g_{\text{eff}} \gg 1$ the supergravity description is valid. We note also that the expression for dilaton in (4.2) can be recast to

$$e^{\phi} = \frac{1}{N} \frac{g_{\text{eff}}^{(7-p)/2}}{\left(1 + \frac{r^2 (n^T M^T M n) U^{7-p}}{R^{7-p}}\right)^{1/2}}. \tag{4.4}$$

Keeping g_{eff} and r fixed we see from (4.4) that $e^{\phi} \sim 1/N$. Therefore the string loop expansion corresponds to $1/N$ expansion of non-commutative gauge theory.

Since the scalar curvature is r-independent, as far as the effective gauge coupling is concerned, the situation is the same as ordinary brane solution. But since the dilaton is r-dependent this will change the phase structure of the theory. In particular at given fixed energy the effective string coupling will change with r. There is, in fact, a critical length $r_c = g_{YM}\sqrt{N}/bU^{(7-p)/2}$ in which for $r \gg r_c$ the non-commutative effects become important and the effective string coupling becomes

$$e^{2\phi} \sim \frac{(g_{YM}^2 N)^{(9-p)/2}}{N^2} \frac{U^{(p-5)(7-p)/2}}{b^2 r^2}, \tag{4.5}$$

where $b^2 = n^T M^T M n$ is twist parameter. Therefore the effective string coupling decreases for large distance and the gravity description becomes more applicable. On the other hand, at given fixed energy, the non-commutative effects become less important for distances smaller than the critical length r_c.

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One can also study the phase structure of the theory which is very similar to that in non-commutative field theory with constant non-commutative parameter. The only difference is that the distinguished points where the description of the theory has to be changed is now r-dependent.

In the notation of [16] the dimensionless effective non-commutative parameter is given by

$$a^{\rm eff} = \left(\frac{rbU^2}{g_{\rm eff}}\right)^{\frac{2}{7-p}} = \left(\frac{r}{r_c}\right)^{\frac{2}{7-p}} . \tag{4.6}$$

At small distances $r \ll r_c$ the non-commutative effects are small and the effective description of the worldvolume theory is in terms of a commutative field theory. Note that this distance is energy-dependent (U-dependent) which means it changes with energy.

Form the expression of the dimensionless effective non-commutative parameter (4.6) one can read the non-commutative parameter seen by the gauge theory. In fact we get

$$[x_p, x_i] \sim rb, \tag{4.7}$$

while in the polar coordinates it may be written as $[x_p, \theta] = b$ in agreement with [57].

To understand these theories better it is worth to study some of them in more detail.

4.1 D3-brane

This case has recently been studied in [57]. Here, just for completeness, we will review this case again. In our notation the corresponding matrix M and unit vector n are given by

$$M = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}, \qquad n^T = (\cos\theta \quad \sin\theta). \tag{4.8}$$

Plugging these into the general solution (4.2) we find

$$\begin{aligned} l_s^{-2} ds^2 &= \left(\frac{U}{R}\right)^2 \left(-dt^2 + dr^2 + \frac{dx_3^2 + r^2 d\theta^2}{1 + \frac{r^2 b^2 U^4}{R^4}}\right) + \left(\frac{R}{U}\right)^2 \left(dU^2 + U^2 d\Omega_5^2\right), \\ e^{2\phi} &= \frac{\bar{g}_s^2}{1 + \frac{r^2 b^2 U^4}{R^4}}, \qquad B_{3\theta} = \alpha' \frac{r^2 b \frac{U^4}{R^4}}{1 + \frac{r^2 b^2 U^4}{R^4}}, \qquad C_{0r} = \frac{\alpha'}{\bar{g}_s} br \frac{U^4}{R^4}, \end{aligned} \tag{4.9}$$

where $R^4 = 2 g_{YM}^2 N$. We have also a RR 4-form corresponding to the original N D3-branes which is given by $dC_4 = \frac{1}{\bar{g}_s} N l_s^5 \epsilon_5$ where ϵ_5 is the worldvolume of the 5-sphere.

In spirit of AdS/CFT correspondence one may suspect that type-IIB string theory in this background is dual to a non-commutative field theory with non-constant non-commutative parameter. The field content of the theory is the same as $\mathcal{N} = 4$ SYN theory in four dimension though the theory is not supersymmetric.

4.2 D4-brane

Let us consider D4-brane wrapping a Melvin universe. Using our general procedure the corresponding solution is given by (4.2) with the following matrix M and unit vector n

$$M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & b \\ 0 & -b & 0 \end{pmatrix}, \qquad n^T = (\cos\theta \quad \sin\theta\, \cos\phi \quad \sin\theta\, \sin\phi). \tag{4.10}$$

Since the matrix M is block diagonal one may work in the reduced subspace to simplify the computations. To do that we consider a twist such that x_1 remains untouched, while x_2 and x_3 transform the same as D3-brane case. Therefore we consider a twist which acts as follows

$$(dx_1 \;\; dx_2 \;\; dx_3) = (dx_1 \;\; dx_2 + bx_3 dx_4 \quad dx_3 - bx_2 dx_4). \tag{4.11}$$

In this notation we have $(x_2 \quad x_3) = r(\cos\theta \quad \sin\theta)$ and the supergravity solution reads

$$\begin{aligned} l_s^{-2} ds^2 &= \left(\frac{U}{R}\right)^{3/2} \left(-dt^2 + dx_1^2 + dr^2 + \frac{dx_4^2 + r^2 d\theta^2}{1 + \frac{r^2 b^2 U^3}{R^3}}\right) + \left(\frac{R}{U}\right)^{3/2} \left(dU^2 + U^2 d\Omega_4^2\right), \\ e^{2\phi} &= \bar{g}_s^2 \frac{(U/R)^{3/2}}{1 + \frac{r^2 b^2 U^3}{R^3}}, \qquad B_{4\theta} = \alpha' \frac{r^2 b \frac{U^3}{R^3}}{1 + \frac{r^2 b^2 U^3}{R^3}}, \qquad C_{01r} = \frac{\alpha'^{3/2}}{\bar{g}_s} br \frac{U^3}{R^3}, \end{aligned} \tag{4.12}$$

where $R^3 = g_{YM}^2 N/4\pi$. There is also a RR 5-form representing the original N D4-branes.

Actually this is the solution which could be obtained by using T-duality from D3-brane solution (4.9). This solution could provide the gravity description of a gauge theory in five dimensions whose field content is the same as five dimensional SYM theory with 16 supercharges, though the supersymmetry is broken because of non-zero B-field. One can then use this supergravity solution to study the phase structure of the theory. In fact it can be seen that the phase structure is very similar to the non-commutative field theory with constant non-commutative parameter. In particular the dilaton is small both in IR and UV limits and therefore in both limits the type-IIA supergravity solution provides a good description for the theory.

4.3 D5-brane

In D5-brane case we recognize two different cases. The first case can simply be obtained from the D4-brane solution (4.12) by a T-duality in a transverse direction to the brane. In this case the supergravity solution reads

$$\begin{aligned} l_s^{-2} ds^2 &= \frac{U}{R} \left(-dt^2 + dx_1^2 + dx_2^2 + dr^2 + \frac{dx_5^2 + r^2 d\theta^2}{1 + \frac{r^2 b^2 U^2}{R^2}}\right) + \frac{R}{U} \left(dU^2 + U^2 d\Omega_3^2\right), \\ e^{2\phi} &= \bar{g}_s^2 \frac{\frac{U^2}{R^2}}{1 + \frac{r^2 b^2 U^2}{R^2}}, \qquad B_{5\theta} = \alpha' \frac{r^2 b \frac{U^2}{R^2}}{1 + \frac{r^2 b^2 U^2}{R^2}}, \qquad C_{012r} = \frac{\alpha'^2}{\bar{g}_s} br \frac{U^2}{R^2}, \end{aligned} \tag{4.13}$$

where $R^2 = N\bar{g}_s$. We have also a RR 2-form corresponding to N D5-branes.

On the other hand we can consider the most general case where all coordinates in the worldvolume of the brane are touched by the twist. In this case the most general form for matrix M up to a $so(4)$ transformation is given by

$$M = \begin{pmatrix} 0 & b & 0 & 0 \\ -b & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 \end{pmatrix}, \tag{4.14}$$

and since the matrix is block diagonal the computations become simpler if we parametrize the unit vector n as follows

$$n^T = (\sin\theta\ \cos\phi \quad \cos\theta\ \cos\phi \quad \sin\psi\ \sin\phi \quad \cos\psi\ \sin\phi\,)\,. \tag{4.15}$$

With this parametrization the supergravity solution (4.2) reads

$$\begin{aligned}
l_s^{-2}ds^2 &= \frac{U}{R}\left(-dt^2+dr^2+r^2d\tilde{\Omega}_3^2+\frac{dx_5^2-r^4b^2\frac{U^2}{R^2}(\cos^2\phi d\theta+\sin^2\phi d\psi)^2}{1+\frac{r^2b^2U^2}{R^2}}\right)\\
&\quad+\frac{R}{U}\left(d\rho^2+\rho^2d\Omega_3\right),\\
e^{2\phi} &= \bar{g}_s^2\frac{\frac{U^2}{R^2}}{1+\frac{r^2b^2U^2}{R^2}},\qquad \sum_i B_{pi}d\theta_i=\frac{r^2b\frac{U^2}{R^2}}{1+\frac{r^2b^2U^2}{R^2}}(\cos^2\phi d\theta+\sin^2\phi d\psi),
\end{aligned} \tag{4.16}$$

where $d\tilde{\Omega}_3^2 = d\phi^2+\cos^2\phi d\theta^2+\sin^2\phi d\psi^2$. Beside the RR 2-form representing the original N D5-branes there is also a RR 4-form which in the original x_i coordinates is given by

$$C_4 = -b\frac{U^2}{R^2}dt\wedge\Big((x_1dx_1+x_2dx_2)\wedge dx_3\wedge dx_4+dx_1\wedge dx_2\wedge(x_3dx_3+x_4dx_4).\Big) \tag{4.17}$$

This supergravity solution provides a dual description of a six dimensional non-commutative gauge theory with non-constant non-commutative parameter whose field content is the same as that in six dimensional SYM with 16 supercharges which is defined in the worldvolume of D5-brane, though because of non-constant B-field the supersymmetry is broken. One can then use this supergravity solution to study the corresponding non-commutative gauge theory. In fact the phase digram of this system is very similar to one with constant non-commutative parameter, though the typical scales where we will have to change our description are r-dependent. At IR limit we expect that the non-commutative effects become negligible, and therefore for $U \ll \sqrt{g_{YM}N}/rb$ the good description is given by D5-brane solution without B-field and its phase digram would be the same as six dimensional supersymmetric gauge theory with 16 supercharges. On the other hand at UV limit where the effects of non-commutativity become important one needs to take into account the whole solution. In particular in this limit the dilaton behaves like $e^{\phi} \sim \bar{g}_s/rb$ and we can trust the gravity description as far as $\bar{g}_s \ll rb$, otherwise we need to make an S-duality and work in S-dual picture. In this case we will have to deal with type-IIB NS5-brane in the presence of RR field which is the subject of the next section.

5. NS5-brane wrapping a Melvin universe

In this section we will study type-II NS5-brane wrapping a Melvin universe. This will lead to the supergravity solution of NS5-brane in the presence of several RR-fields. This might be thought of as new deformation of the theories which live in the worldvolume of NS5-brane in the presence of RR-field. When the deformation parameter is constant, these theories are known as ODp-theories which include open Dp-branes. On the other hand in the case we are interested in the deformation parameter is not constant and therefore we get new theories and this is the aim of this section to study these theories using their dual supergravity solutions.

These supergravity solutions can be obtained from D5-brane using a chain of S and T dualities. As we saw in the previous section depending on how the twist acts on the coordinates, there are two different deformations of type-IIB D5-brane. Starting from these solutions and apply S-duality one can find type-IIB NS5-brane in the presence of RR 2-form.[2] The simplest case is when the twist acts only on two coordinates (4.13). Then a series of T-duality will generate other possible RR fields. Doing so one finds

$$\begin{aligned}
ds^2 &= (1+r^2b^2f^{-1}/g_s^2\alpha'^2)^{1/2}\Big[-dt^2+dr^2+\sum_{i=1}^{p-1}dx_i^2+\frac{\sum_{a=p}^4 dy_a^2}{1+r^2b^2f^{-1}/g_s^2\alpha'^2}\\
&\qquad\qquad +f(d\rho^2+\rho^2 d\Omega_3^2)\Big],\\
C_{1\cdots(5-p)} &\sim \frac{1}{g_s}\frac{r^2bf^{-1}}{1+r^2b^2f^{-1}/g_s^1\alpha'^2},\qquad C_{0r1\cdots(p-1)}\sim\frac{1}{g_s}brf^{-1},\\
e^{2\phi} &= g_s^2(1+r^2b^2f^{-1}/g_s^2\alpha'^2)^{(p-1)/2}\frac{f}{r^{2\delta_{p5}}}.
\end{aligned}\tag{5.1}$$

Here in order to unify the solutions we have used a notation in which $dy_4 = rd\theta$, $dx_4 = d\theta/r$. We have also a non-zero B-field representing the original N NS5-branes which in our notation is give by $dB = Nl_s^2\epsilon_3$ with ϵ_3 being volume of the 3-sphere.

We can also consider the decoupling limit of these solutions. The corresponding decoupling limit can be obtained from decoupling limit of D5-brane using S-duality in which $l_s^2 \to g_s l_s^2$ and $g_s \to g_s^{-1}$. Thus the decoupling limit of the above supergravity solutions is defined as a limit in which $g_s \to 0$ while keeping the following quantities fixed

$$U=\frac{\rho}{g_s l_s^2},\qquad l_s=\text{fixed}.\tag{5.2}$$

which is the same as the decoupling limit of little string theory [6]. Note that to make U of dimension of energy, we have also added l_s in the definition of U. In this limit the supergravity solutions (5.1) read

$$\begin{aligned}
ds^2 &= \left(1+\frac{r^2b^2U^2}{N\alpha'}\right)^{1/2}\left[-dt^2+dr^2+\sum_{i=1}^{p-1}dx_i^2+\frac{\sum_{a=p}^4 dy_a^2}{1+\frac{r^2b^2U^2}{N\alpha'}},+\frac{N\alpha'}{\rho^2}(d\rho^2+\rho^2d\Omega_3^2)\right],\\
e^{2\phi} &= \frac{N}{\alpha'U^2}\left(1+\frac{r^2b^2U^2}{N\alpha'}\right)^{(p-1)/2},\qquad C_{0r1\cdots(p-1)}=\frac{\alpha'^{(p+1}/2}{g_s}\frac{brU^2}{N\alpha'},\\
C_{1\cdots(5-p)} &= \frac{\alpha'^{(5-p)/2}}{g_s}\frac{\frac{r^2bU^2}{N\alpha'}}{1+\frac{r^2b^2U^2}{N\alpha'}}.
\end{aligned}\tag{5.3}$$

These supergravity solutions should be compared with supergravity solutions describing ODp-theories which are given by NS5-brane in the presence of RR p-form. Here we also

[2] Under S-duality we have $\phi \to -\phi$ and $ds^2 \to e^{-\phi}ds^2$ where ds^2 is the metric in string frame. Moreover the NSNS B-field gets changed to the RR 2-form.

have the same structure though the RR fields are also a function of the brane worldvolume coordinate r. This would result that the corresponding quantum theories should be deformed by a non-constant parameter.

It is also constructive to study transverse gravitons scattering from these NS5-brane solutions in the presence of r-dependent RR fields. Ultimately this leads to a Schrördinger-like equation with the following potential

$$V(\eta) = -1 + \left(\frac{3}{4} - 1\right) \frac{N\omega^2\alpha'}{\eta^2}, \tag{5.4}$$

where $\eta = \omega\rho$ is a dimensionless radial coordinate. Following [66] one may conclude that the theory has a mass gap of order of $m_{gap} \sim 1/\sqrt{N\alpha'}$ which is exactly the same as six dimensional theories live on type-II NS5-brane.

6. M5-brane wrapping a Melvin universe and its descendant theories

In this section we shall study supergravity solution of M5-brane wrapping 11-dimensional Melvin universe. This would give a new non-commutative deformation of (0,2) theory with non-constant non-commutative parameter. From gravity point of view this corresponds to the case where we have M5-brane in the presence of M-theory 3-form which depends on the coordinates of the M5-brane worldvolume. In the case where the 3-form was independent of the brane worldvolume coordinates, it was shown [27] that one could consider a decoupling limit such that the theory in the M5-brane worldvolume decouples from bulk gravity and the decoupled theory has light open membrane. This theory is called OM theory.

Upon compactifying OM theory on a circle and using a chain of T-duality, we will get supergravity solutions of Dp-brane in the presence of electric E-field (B-field in the time direction). It was also shown [23 − 27] unitarity implies that this theory is not a simple gauge theory and in fact the theory on the corresponding worldvolume is indeed a non-commutative version of open string theory. This is the aim of this section to generalize the above construction for the case where the 3-form in M-theory and thereby the E-field in type-II string theories depends on the brane worldvolume coordinates.

Let us first obtain the supergravity solution of M5-brane wrapping an eleven dimensional Melvin universe. This can be done by making use of the type-IIA supergravity solution we have found. Starting from D4-brane or type-IIA NS5-brane one may uplift the solution to find the M5-brane solution. In general a type-IIA supergravity solution representing by 10-dimensional metric, ds^2_{10}, RR one-form and dilaton can be uplifted into 11-dimensional solution whose metric is given by

$$ds^2_{11} = e^{4\phi/3}(dx_{11} + A_\mu dx^\mu)^2 + e^{-2\phi/3} ds^2_{10}. \tag{6.1}$$

Both RR 3-form and B-field under this uplifting go into M-theory 3-forms.

Therefore to find the supergravity solution of M5-brane wrapping a Melvin universe, we can start from D4-brane solution in (3.6) and then uplifting it to 11-dimensional supergravity and sending the radius of $11th$ direction to infinity $R_{11} \to \infty$. In this limit keeping

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$bR_{11} = L$ fixed, one finds

$$\begin{aligned} ds^2 &= (1+r^2L^2f^{-1}/l_p^6)^{1/3}\left[f^{-1/3}\left(-dt^2+dr^2+dx^2+\frac{dy^2+dz^2+r^2d\theta^2}{1+r^2L^2f^{-1}/l_p^6}\right)\right. \\ &\qquad\qquad\qquad\qquad \left.+f^{2/3}(d\rho^2+\rho^2d\Omega_4^2)\right], \\ C_{yz\theta} &= \frac{r^2Lf^{-1}}{1+r^2L^2f^{-1}/l_p^6}, \qquad C_{0rx} = Lrf^{-1}, \qquad f = 1+\frac{\pi N l_p^3}{\rho^3}, \end{aligned} \tag{6.2}$$

where l_p is 11-dimensional Plank scale. There is also a 6-form (magnetic dual to 3-form) representing the original N M5-brane we started with. The decoupling limit of the solution is defined by $l_p \to 0$ while keeping $U = r/l_p^3$ fixed. One can easily write done the supergravity solution in this limit which will provide the gravity description of non-commutative (0,2) theory with non-constant non-commutative parameter. One can also evaluate the curvature of the supergravity solution

$$l_p^2\mathcal{R} \sim \frac{1}{N^{2/3}}\frac{1}{(1+\frac{r^2L^2U^3}{\pi N})^{1/3}}, \tag{6.3}$$

which shows that we can trust the supergravity solution for large-N. The supergravity can also be trusted for scales much more larger that critical length defined in previous section.

One can easily check that upon compactifying this solution on one of a circle of $d\Omega_4^2$ we will end up with type-IIA NS5-brane solution given in (5.1) for $p = 2$. It is also possible to compactify it on other directions to find new solutions in type-IIA string theory. One then use T-duality to find new solutions in type-II string theories. Probably more interesting cases can be found by compactifying the M-theory solution on x or θ and then using a chain S and T-dualities. For example compactifying on x one finds

$$\begin{aligned} ds^2 &= (1+r^2b^2f^{-1}/\alpha'^2)^{1/2}\left[f^{-1/2}\left(-dt^2+dr^2+\frac{dy^2+dz^2+r^2d\theta^2}{1+r^2b^2f^{-1}/\alpha'^2}\right)\right. \\ &\qquad\qquad\qquad\qquad \left.+f^{1/2}(d\rho^2+\rho^2d\Omega_4^2)\right], \\ e^{2\phi} &= g_s^2f^{-1/2}(1+r^2b^2f^{-1}/\alpha'^2)^{1/2}, \qquad B_{0r} = brf^{-1}, \\ C_{yz\theta} &= \frac{1}{g_s}\frac{r^2bf^{-1}}{1+r^2b^2f^{-1}/\alpha'^2}, \qquad f = 1+\frac{\pi N g_s l_s^3}{\rho^3}. \end{aligned} \tag{6.4}$$

This solution represent D4-brane supergravity solution in the presence of non-zero, non-constant E-field (B-field with one leg along the time direction). One can now proceed to find new solutions using T-duality. In fact T-dualizing the solution along y we obtain D3-brane solution smeared in one dimension. Then we can write down the localized D3-brane solution as follows

$$\begin{aligned} ds^2 &= (1+r^2b^2f^{-1}/\alpha'^2)^{1/2}\left[f^{-1/2}\left(-dt^2+dr^2+\frac{dz^2+r^2d\theta^2}{1+r^2b^2f^{-1}/\alpha'^2}\right)+f^{1/2}(d\rho^2+\rho^2d\Omega_5^2)\right], \\ e^{2\phi} &= g_s^2(1+r^2b^2f^{-1}/\alpha'^2), \qquad B_{0r} = brf^{-1}, \\ C_{z\theta} &= \frac{1}{g_s}\frac{r^2bf^{-1}}{1+r^2b^2f^{-1}/\alpha'^2}, \qquad f = 1+\frac{4\pi N g_s l_s^4}{\rho^4}. \end{aligned} \tag{6.5}$$

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Doing the same in z direction we arrive at the following D2-brane solution[3]

$$\begin{aligned}
ds^2 &= (1+r^2b^2f^{-1}/\alpha'^2)^{1/2}\left[f^{-1/2}\left(-dt^2+dr^2+\frac{r^2d\theta^2}{1+r^2b^2f^{-1}/\alpha'^2}\right)+f^{1/2}(d\rho^2+\rho^2d\Omega_6^2)\right],\\
e^{2\phi} &= g_s^2f^{1/2}(1+r^2b^2f^{-1}/\alpha'^2)^{3/2}, \qquad B_{0r}=brf^{-1},\\
C_\theta &= \frac{1}{g_s}\frac{r^2bf^{-1}}{1+r^2b^2f^{-1}/\alpha'^2}, \qquad\qquad f=1+\frac{c_2Ng_sl_s^5}{\rho^5}.
\end{aligned}\tag{6.6}$$

Finally one could perform a T-duality along θ direction to find D1-brane solution as follows

$$\begin{aligned}
ds^2 &= (1+r^2b^2f^{-1}/\alpha'^2)^{1/2}\left[f^{-1/2}(-dt^2+dr^2)+f^{1/2}(d\rho^2+\rho^2d\Omega_7^2)\right],\\
e^{2\phi} &= g_s^2\frac{f}{r^2}(1+r^2b^2f^{-1}/\alpha'^2)^2, \qquad B_{0r}=brf^{-1},\\
\chi &= \frac{1}{g_s}\frac{r^2bf^{-1}}{1+r^2b^2f^{-1}/\alpha'^2}, \qquad\qquad f=1+\frac{c_1Ng_sl_s^6}{\rho^6}.
\end{aligned}\tag{6.7}$$

where χ is type-IIB RR scaler. In all solutions we have an extra RR p-field representing the original N Dp-branes.

It is also possible to perform a T-duality along a direction transverse to the brane in the solution (6.4). Doing so we find a D5-brane solution in the presence of E-field which could also be obtained from solution (5.1) for $p=1$ using S-duality.

These solutions, upon taking decoupling limit, must be compared with non-commutative open string theory [23 − 27]. We note, however, that the decoupling limit of these solutions is the same as the case when we have B-field, namely $l_s\rightarrow 0$ while $g_sl_s^{p-3}$ fixed.

One the other hand in the case of constant non-commutative parameter, having E-field would cause to have non-commutativity in the time direction and theory would be ill-defined unless we add open string in the game. This was automatically the case by taking near critical E-field. But in our case, at least as far as the supergravity is concerned, the decoupling limit is the same as the one with B-field.

If in this case the theory is going to be a non-commutative theory with non-commutative time, one might suspect that the dual theory is not unitary, unless we could add open string in the game. Otherwise, the theory would be ill-defined and from gravity point of view this could mean that the gravity solutions are unstable. It would be interesting to study this case in more detail.

Finally we note that if we compactify the solution on θ direction one finds

$$\begin{aligned}
ds^2 &= r\left[f^{-1/2}\left(-dt^2+dr^2+dx^2+\frac{dy^2+dz^2}{1+r^2b^2f^{-1}/\alpha'^2}\right)+f^{1/2}(d\rho^2+\rho^2d\Omega_4^2)\right],\\
e^{2\phi} &= g_s^2\frac{r^3f^{-1/2}}{1+r^2b^2f^{-1}/\alpha'^2} \quad B_{yz}=\frac{r^2bf^{-1}}{1+r^2b^2f^{-1}/\alpha'^2}, \quad C_{0rx}=\frac{1}{g_s}brf^{-1}.
\end{aligned}\tag{6.8}$$

Performing a T-duality along x direction we will get another D3-brane solution in the presence of non-zero non-uniform B-field as follows

$$ds^2 = r\left[f^{-1/2}\left(-dt^2+dr^2+\frac{dy^2+dz^2}{1+r^2b^2f^{-1}/\alpha'^2}\right)+f^{1/2}(d\rho^2+\rho^2d\Omega_4^2)\right],$$

[3] This solution can also be uplifted into M-theory to get M2-brane in the presence of non-constant 3-form with two legs along the brane worldvolume and one leg along the transever direction to the brane. This could be used to study a new deformation of 3-dimension $\mathcal{N}=8$ SCFT.

$$e^{2\phi} = g_s^2 \frac{r^2}{1+r^2b^2f^{-1}/\alpha'^2} \quad B_{yz} = \frac{r^2bf^{-1}}{1+r^2b^2f^{-1}/\alpha'^2}, \quad C_{0r} = \frac{1}{g_s}brf^{-1}. \tag{6.9}$$

This solution upon taking the decoupling limit will provide the gravity description for a non-commutative gauge theory with non-constant non-commutative parameter which should not be the same as what we have reviewed in section 2.

7. Light-like twist

In this section, for completeness, we will study Dp-brane wrapping a background with light-like twist. The supergravity solution of light-like twist has been considered in [37]. In this section we generalize this construction for the case where the light-like B-field depends on the brane worldvolume coordinates.

To find the corresponding solution we start from the general solution (3.6) and consider the following boost in the x_p direction

$$\hat{t} = \cosh\gamma\, t - \sinh\gamma\, x_p, \qquad \hat{x}_p = -\sinh\gamma\, t - \cosh\gamma\, x_p\,, \tag{7.1}$$

or

$$x^+ = e^{-\gamma}y^+, \qquad x^- = e^{\gamma}y^-\,, \tag{7.2}$$

with $y^\pm = x_p \pm t$ and $x^\pm = \hat{x}_p \pm \hat{t}$.

To have a light-like limit we now take the infinite boost limit, $\gamma \to \infty$. In order to end up with a light-like limit vector with finite component we must simultaneously scale $M \to 0$ while $Me^\gamma = \tilde{M}$ is kept fixed. In this limit the background (3.6) reads

$$\begin{aligned} ds^2 &= f^{-1/2}\left(-4dx^+dx^- - \frac{r^2f^{-1}}{\alpha'^2}(n^T\tilde{M}^T\tilde{M}n)(dx^+)^2 + dr^2 + r^2dn^Tdn\right) \\ &\quad + f^{1/2}\left(dU^2 + U^2d\Omega_{8-p}^2\right), \\ e^{2\phi} &= g_s^2f^{(3-p)/2}, \qquad \sum_i B_{+i}dx_i = r^2f^{-1}(dn^T\tilde{M}n). \end{aligned} \tag{7.3}$$

We have also an extra RR p-form representing N Dp-branes.

Similarly we can also apply this procedure for other solutions we have found in this paper. Upon taking the decoupling limit these solution would provide the supergravity description for different theories, in various dimensions with light-like non-commutative deformation with non-constant non-commutative parameter.

8. Discussions

In this paper we have obtained supergravity solutions of different branes in type-II string theories and M-theory wrapping a Melvin universe. Practically these solutions can be obtained by a chain of T and S dualities and twists. These supergravity solutions correspond to Dp-brane in presence of non-zero B-field along its worldvolume such that the B-field depends on the brane worldvolume coordinates (non-constant). Doing the same procedure

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for NS5-brane we have found a class of supergravity solutions corresponding to type-II NS5-branes in the presence of different RR fields along the brane worldvolume which are coordinates dependent.

The supergravity solution of M5-brane in the presence of non-zero, non-constant 3-form along the worldvolume of the brane has also been obtained. Upon compactifying this solution on a circle, depending on which direction is taken to be compact, and also using a chain of T and S dualities we have been able to find new supergravity solutions corresponding to different brane solutions in presence of B-field with one leg along time direction. We have also considered a light-like B-field/RR field which can be obtained from the solutions we have studied in the previous sections by making use of an infinite boost.

We have seen that there is a limit in which the worldvolume theory of these solutions decouples from the bulk gravity and therefore they could provide supergravity description of new deformation of the brane worldvolume theory. In fact the situation is very similar to the case when we have Dp/NS5/M5 branes in the presence of non-zero B-field/RR field/3-form which was independent of the brane worldvolume coordinates (constant). The worldvolume theory decouples from the bulk and therefore would provide a gravity description of non-commutative gauge theory/ODp-theory/OM-theory. It is also known that upon compcatifying OM-theory on a circle and using T and S dualities one will get NCOS theory. From supergravity point of view this corresponds to Dp-brane in the presence of E-field.

Therefore one may conclude that the supergravity solution we have found in this paper would provide the gravity description of non-commutative deformation of the corresponding theories where the non-commutative parameter is non-constant.

In general we would expect that by turning on a non-zero, non-constant B-field in the worldvolume on Dp-brane, the worldvolume theory deforms to a non-commutative gauge theory with non-constant non-commutativity parameter. Because of non-constant B-field these theories are not supersymmetric, nevertheless the field content of them are the same as their undeform supersymmetric theories. In IR limit where the non-commutative effects are negligible, the supersymmetry restores. Since the theory is not supersymmetric, one may wonder if the corresponding supergravity solution is stable. This is a point one needs to be check, though we have not studied it in this paper.

One interesting feature of non-commutative field theories with non-constant parameter is that they have a natural critical length which controls the effects of non-commutativity. As we have seen, from supergravity description point of view, the non-commutative effects are controlled by a dimensionless parameter given by

$$a^{\rm eff} = \left(\frac{rbU^2}{g_{\rm eff}}\right)^{\frac{2}{7-p}} = \left(\frac{r}{r_c}\right)^{\frac{2}{7-p}} . \tag{8.1}$$

Therefore the non-commutative effects are important at distances which are of order of r_c while they are negligible for distances smaller than this natural length. This is an interesting fact, saying that, we would expect to see non-commutative effects at large scaler in contrast to our standard intuition and what we have learned in the case of non-commutative field theory with constant parameter where we expect to see the effects at short distances.

To be concrete let us consider the D3-brane case in more detail. In fact an interesting feature about the critical length, r_c, is that it is a function of U, namely

$$r_c = \frac{\sqrt{g_s N}}{bU^2}. \tag{8.2}$$

According to AdS/CFT correspondence one may think about coordinate U as the scale of energy and therefore the critical length is energy dependent parameter. At any given fixed energy, the non-commutative effects are given in terms of critical length (8.2). On the other hand if we consider an s-wave scaler field Φ with frequency ω in the background (4.9) the wave equation is

$$U^{-3}\partial_U(U^{-5}\partial_U\Phi) + \omega^2 \frac{g_{YM}^2 N}{U^2}\Phi = 0, \tag{8.3}$$

which shows that the solution only depends on $\omega^2 \frac{g_{YM}^2 N}{U^2}$ and so the radial dependence of the solution has the holographic relation with energy. Actually this means that a UV cutoff U on radius of AdS_5 translates into a UV cutoff $\mathcal{E}$ in the dual CFT, such that [67, 68]

$$\mathcal{E} = \frac{U}{\sqrt{g_s N}}. \tag{8.4}$$

Alternatively one would say that at the energy scale U in the bulk, only those modes in CFT will be excited which are in region given by $\delta X = \frac{\sqrt{g_s N}}{U}$ which is very similar to the relation we get for critical length. In fact these two can be combined to get a limit on the non-commutativity

$$r_c = \frac{1}{\sqrt{g_s N}} \frac{(\delta X)^2}{b}, \tag{8.5}$$

where δX would be a typical length of our normal life which could be of order of meter. On the other hand the non-commutativity effects are important in the distances of order of r_c and since we have not seen these effects so far, therefore at least r_c must be of order of a typical cosmological length, or the radius of the world which is of order of 10^{26} m. Putting this information as an input one may put a bound on b or to be precise on $\sqrt{g_s} b \sim 10^{-27}$ m if we assume N is of order of 10^2.

To summarize we note that the non-commutative effects with non-constant parameter could affect the long distance physics and therefore might be relevant in cosmology. It would be interesting to study a cosmological model with such a non-commutativity and would probably put a bound on the non-commutative parameter using WMAP data.

To understand the feature of this kind of non-commutative field theory one could also study other objects in this theory like Wilson loop, monopoles and other salitonic solutions using AdS/CFT correspondence. In particular one can use the open string action in this background to study Wilson loop and thereby the effective potential between the external objects like "quarks" following [69, 70]. In fact in this case the situation is very similar to the case where the non-commutative parameter was constant and actually we get the same expression as what studied in [16] except that now the final results are r-dependent.

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References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[2] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109].

[3] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[4] M. Alishahiha, H. Ita and Y. Oz, *Graviton scattering on D6 branes with B fields*, *JHEP* **06** (2000) 002 [hep-th/0004011].

[5] N. Itzhaki, J.M. Maldacena, J. Sonnenschein and S. Yankielowicz, *Supergravity and the large-N limit of theories with sixteen supercharges*, *Phys. Rev.* **D 58** (1998) 046004 [hep-th/9802042].

[6] O. Aharony, M. Berkooz, D. Kutasov and N. Seiberg, *Linear dilatons, NS5-branes and holography*, *JHEP* **10** (1998) 004 [hep-th/9808149].

[7] S.R. Das and S.P. Trivedi, *Three brane action and the correspondence between $N = 4$ Yang-Mills theory and anti de Sitter space*, *Phys. Lett.* **B 445** (1998) 142 [hep-th/9804149].

[8] S. Ferrara, M.A. Lledo and A. Zaffaroni, *Born-Infeld corrections to D3 brane action in $AdS_5 \times S^5$ and $N = 4$, $D = 4$ primary superfields*, *Phys. Rev.* **D 58** (1998) 105029 [hep-th/9805082].

[9] M.R. Douglas and C.M. Hull, *D-branes and the noncommutative torus*, *JHEP* **02** (1998) 008 [hep-th/9711165].

[10] F. Ardalan, H. Arfaei and M.M. Sheikh-Jabbari, *Noncommutative geometry from strings and branes*, *JHEP* **02** (1999) 016 [hep-th/9810072].

[11] C.-S. Chu and P.-M. Ho, *Noncommutative open string and D-brane*, *Nucl. Phys.* **B 550** (1999) 151 [hep-th/9812219].

[12] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *JHEP* **09** (1999) 032 [hep-th/9908142].

[13] A. Hashimoto and N. Itzhaki, *Non-commutative Yang-Mills and the AdS/CFT correspondence*, *Phys. Lett.* **B 465** (1999) 142 [hep-th/9907166].

[14] J.M. Maldacena and J.G. Russo, *Large-N limit of non-commutative gauge theories*, *JHEP* **09** (1999) 025 [hep-th/9908134].

[15] M. Li and Y.-S. Wu, *Holography and noncommutative Yang-Mills*, *Phys. Rev. Lett.* **84** (2000) 2084 [hep-th/9909085].

[16] M. Alishahiha, Y. Oz and M.M. Sheikh-Jabbari, *Supergravity and large-N noncommutative field theories*, *JHEP* **11** (1999) 007 [hep-th/9909215].

[17] T. Harmark and N.A. Obers, *Phase structure of non-commutative field theories and spinning brane bound states*, *JHEP* **03** (2000) 024 [hep-th/9911169].

[18] J.L.F. Barbon and E. Rabinovici, *On 1/n corrections to the entropy of noncommutative Yang- Mills theories*, *JHEP* **12** (1999) 017 [hep-th/9910019].

[19] J.X. Lu and S. Roy, *(p+1)-dimensional noncommutative Yang-Mills and D$(p-2)$ branes*, *Nucl. Phys.* **B 579** (2000) 229 [hep-th/9912165].

[20] R.-G. Cai and N. Ohta, *Noncommutative and ordinary super Yang-Mills on (D$(p-2)$,Dp) bound states*, *JHEP* **03** (2000) 009 [hep-th/0001213].

[21] R.-G. Cai and N. Ohta, *On the thermodynamics of large-N non-commutative super Yang-Mills theory*, *Phys. Rev.* **D 61** (2000) 124012 [hep-th/9910092].

[22] R.-G. Cai and N. Ohta, *(f1, d1, d3) bound state, its scaling limits and* $\mathrm{SL}(2,\mathbb{Z})$ *duality*, *Prog. Theor. Phys.* **104** (2000) 1073 [hep-th/0007106].

[23] N. Seiberg, L. Susskind and N. Toumbas, *Strings in background electric field, space/time noncommutativity and a new noncritical string theory*, *JHEP* **06** (2000) 021 [hep-th/0005040].

[24] R. Gopakumar, J.M. Maldacena, S. Minwalla and A. Strominger, *S-duality and noncommutative gauge theory*, *JHEP* **06** (2000) 036 [hep-th/0005048].

[25] J.L.F. Barbon and E. Rabinovici, *Stringy fuzziness as the custodian of time-space noncommutativity*, *Phys. Lett.* **B 486** (2000) 202 [hep-th/0005073].

[26] J. Gomis and T. Mehen, *Space-time noncommutative field theories and unitarity*, *Nucl. Phys.* **B 591** (2000) 265 [hep-th/0005129].

[27] R. Gopakumar, S. Minwalla, N. Seiberg and A. Strominger, *OM theory in diverse dimensions*, *JHEP* **08** (2000) 008 [hep-th/0006062].

[28] O. Aharony, J. Gomis and T. Mehen, *On theories with light-like noncommutativity*, *JHEP* **09** (2000) 023 [hep-th/0006236].

[29] M. Alishahiha, Y. Oz and J.G. Russo, *Supergravity and light-like non-commutativity*, *JHEP* **09** (2000) 002 [hep-th/0007215].

[30] R.-G. Cai and N. Ohta, *Lorentz transformation and light-like noncommutative SYM*, *JHEP* **10** (2000) 036 [hep-th/0008119].

[31] A. Bergman and O.J. Ganor, *Dipoles, twists and noncommutative gauge theory*, *JHEP* **10** (2000) 018 [hep-th/0008030].

[32] S. Chakravarty, K. Dasgupta, O.J. Ganor and G. Rajesh, *Pinned branes and new non Lorentz invariant theories*, *Nucl. Phys.* **B 587** (2000) 228 [hep-th/0002175].

[33] K. Dasgupta, O.J. Ganor and G. Rajesh, *Vector deformations of $N=4$ super- Yang-Mills theory, pinned branes and arched strings*, *JHEP* **04** (2001) 034 [hep-th/0010072].

[34] A. Bergman, K. Dasgupta, O.J. Ganor, J.L. Karczmarek and G. Rajesh, *Nonlocal field theories and their gravity duals*, *Phys. Rev.* **D 65** (2002) 066005 [hep-th/0103090].

[35] K. Dasgupta and M.M. Sheikh-Jabbari, *Noncommutative dipole field theories*, *JHEP* **02** (2002) 002 [hep-th/0112064].

[36] M. Alishahiha and H. Yavartanoo, *Supergravity description of the large-N noncommutative dipole field theories*, *JHEP* **04** (2002) 031 [hep-th/0202131].

[37] M. Alishahiha and O.J. Ganor, *Twisted backgrounds, pp-waves and nonlocal field theories*, *JHEP* **03** (2003) 006 [hep-th/0301080].

[38] T. Harmark, *Open branes in space-time non-commutative little string theory*, *Nucl. Phys.* **B 593** (2001) 76 [hep-th/0007147].

[39] M. Alishahiha, *On type-II ns5-branes in the presence of an RR field*, hep-th/0002198.

[40] P.-M. Ho and Y.-T. Yeh, *Noncommutative D-brane in non-constant NS-NS B field background*, *Phys. Rev. Lett.* **85** (2000) 5523 [hep-th/0005159].

[41] L. Cornalba and R. Schiappa, *Nonassociative star product deformations for D-brane worldvolumes in curved backgrounds*, *Commun. Math. Phys.* **225** (2002) 33 [hep-th/0101219].

[42] P.-M. Ho and S.-P. Miao, *Noncommutative differential calculus for D-brane in non-constant B field background*, *Phys. Rev.* **D 64** (2001) 126002 [hep-th/0105191].

[43] M. Herbst, A. Kling and M. Kreuzer, *Star products from open strings in curved backgrounds*, *JHEP* **09** (2001) 014 [hep-th/0106159].

[44] B.L. Cerchiai, *The Seiberg-Witten map for a time-dependent background*, *JHEP* **06** (2003) 056 [hep-th/0304030].

[45] X. Calmet and M. Wohlgenannt, *Effective field theories on non-commutative space-time*, *Phys. Rev.* **D 68** (2003) 025016 [hep-ph/0305027].

[46] O. Bertolami and L. Guisado, *Noncommutative field theory and violation of translation invariance*, *JHEP* **12** (2003) 013 [hep-th/0306176].

[47] D. Robbins and S. Sethi, *The uv/ir interplay in theories with space-time varying non-commutativity*, *JHEP* **07** (2003) 034 [hep-th/0306193].

[48] W. Behr and A. Sykora, *Construction of gauge theories on curved noncommutative spacetime*, *Nucl. Phys.* **B 698** (2004) 473 [hep-th/0309145].

[49] A. Das and J. Frenkel, *Kontsevich product and gauge invariance*, *Phys. Rev.* **D 69** (2004) 065017 [hep-th/0311243].

[50] V. Gayral, J.M. Gracia-Bondia and F. Ruiz Ruiz, *Position-dependent noncommutative products: classical construction and field theory*, *Nucl. Phys.* **B 727** (2005) 513 [hep-th/0504022].

[51] L. Dolan and C.R. Nappi, *Noncommutativity in a time-dependent background*, *Phys. Lett.* **B 551** (2003) 369 [hep-th/0210030].

[52] A. Hashimoto and S. Sethi, *Holography and string dynamics in time-dependent backgrounds*, *Phys. Rev. Lett.* **89** (2002) 261601 [hep-th/0208126].

[53] D.A. Lowe, H. Nastase and S. Ramgoolam, *Massive IIA string theory and matrix theory compactification*, *Nucl. Phys.* **B 667** (2003) 55 [hep-th/0303173].

[54] A. Hashimoto and K. Thomas, *Dualities, twists and gauge theories with non-constant non-commutativity*, *JHEP* **01** (2005) 033 [hep-th/0410123].

[55] M. Alishahiha and S. Parvizi, *Branes in time-dependent backgrounds and AdS/CFT correspondence*, *JHEP* **10** (2002) 047 [hep-th/0208187].

[56] R.-G. Cai, J.-X. Lu and N. Ohta, *Ncos and D-branes in time-dependent backgrounds*, *Phys. Lett.* **B 551** (2003) 178 [hep-th/0210206].

[57] A. Hashimoto and K. Thomas, *Non-commutative gauge theory on D-branes in Melvin universes*, hep-th/0511197.

[58] M. Kontsevich, *Deformation quantization of poisson manifolds, I*, *Lett. Math. Phys.* **66** (2003) 157 [q-alg/9709040].

[59] T. Takayanagi and T. Uesugi, *D-branes in Melvin background*, *JHEP* **11** (2001) 036 [hep-th/0110200].

[60] T. Takayanagi and T. Uesugi, *Flux stabilization of D-branes in NS-NS Melvin background*, *Phys. Lett.* **B 528** (2002) 156 [hep-th/0112199].

[61] E. Dudas and J. Mourad, *D-branes in string theory Melvin backgrounds*, *Nucl. Phys.* **B 622** (2002) 46 [hep-th/0110186].

[62] J.G. Russo and A.A. Tseytlin, *Supersymmetric fluxbrane intersections and closed string tachyons*, *JHEP* **11** (2001) 065 [hep-th/0110107].

[63] J.G. Russo and A.A. Tseytlin, *Magnetic backgrounds and tachyonic instabilities in closed superstring theory and M-theory*, *Nucl. Phys.* **B 611** (2001) 93 [hep-th/0104238].

[64] M.S. Costa and G. Papadopoulos, *Superstring dualities and p-brane bound states*, *Nucl. Phys.* **B 510** (1998) 217 [hep-th/9612204].

[65] G.T. Horowitz and A. Strominger, *Black strings and P-branes*, *Nucl. Phys.* **B 360** (1991) 197.

[66] S. Minwalla and N. Seiberg, *Comments on the IIA ns5-brane*, *JHEP* **06** (1999) 007 [hep-th/9904142].

[67] L. Susskind and E. Witten, *The holographic bound in anti-de Sitter space*, hep-th/9805114.

[68] A.W. Peet and J. Polchinski, *UV/IR relations in AdS dynamics*, *Phys. Rev.* **D 59** (1999) 065011 [hep-th/9809022].

[69] J.M. Maldacena, *Wilson loops in large-N field theories*, *Phys. Rev. Lett.* **80** (1998) 4859 [hep-th/9803002].

[70] S.-J. Rey and J.-T. Yee, *Macroscopic strings as heavy quarks in large-N gauge theory and anti-de Sitter supergravity*, *Eur. Phys. J.* **C 22** (2001) 379 [hep-th/9803001].

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